PERGAMON

International Journal of Heat and Mass Transfer 42 (1999) 3575–3584

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

# A moving-boundary nodal model for the analysis of the stability of boiling channels

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Received 8 September 1998; received in revised form 4 January 1999

**Abstract**

A moving-boundary nodal model has been derived for the linear and non-linear stability analysis of boiling channels. This model is based on the integration of the conservation (partial differential) equations in space and an approximation of the integral with a weighted average of the integrated variable evaluated at the boundaries of the nodes. The resulting system of ODEs has been used to evaluate the linear stability of a boiling vertical channel. The results obtained with this model, using a relatively small number of nodes, compare favorably with experimental results and calculations obtained with distributed parameter and fixed node models, which require the use of many axial nodes. Supercritical and subcritical Hopf bifurcations have been identified, and the frequency response of the model has been evaluated. These results have been used as the criteria for the determination of the number of single-phase nodes needed for a given frequency range. © 1999 Elsevier Science Ltd. All rights reserved.

## 1. Introduction

Density-wave oscillations may appear in natural circulation boiling systems, such as passive boiling water nuclear reactors (e.g. SBWR). These oscillations may compromise the operational safety of these systems, as well as their mechanical integrity. Given the complexity of such systems it is neither practical nor safe to perform prototypical experiments. A reasonable approach to the study of the stability of boiling systems is the combination of smaller scale experiments and the development of qualified numerical models to simulate the behavior of these systems.

The current understanding of density-wave oscillations is fairly good for linear phenomena (e.g., the onset of instabilities), but is not as well advanced for non-linear phenomena. In particular, limit cycle and chaotic instability modes are not very well understood. Significantly, density-wave oscillations have been observed in operating nuclear reactors, such as those documented during the LaSalle incident [1]. The appearance of density-wave oscillations has also been reported in a pressurized heavy water nuclear reactor (PHWR) [2] under natural circulation after a pump trip. Significantly, the proposed new SBWR operates in natural circulation and is thus inherently prone to density-wave instabilities.

The stability of boiling channels has been studied previously by numerous authors. Ishii [3] presented a simplified analytical criterion for the linear stability of a boiling channel for homogeneous and drift-flux models and established the phase-change number, $N_{pch}$

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PII: S0017-9310(99)00022-8

**Nomenclature**

| | |
|---|---|
| $A_{x-s}$ | cross-sectional flow area |
| $D_H$ | hydraulic diameter |
| $G$ | mass flux |
| $h$ | enthalpy |
| $k$ | thermal conductivity |
| $K$ | hydraulic loss coefficient |
| $L_j$ | length of axial node-j |
| $p$ | pressure |
| $Q_j$ | energy production in axial node-j |
| $q''$ | heat flux |
| $s$ | Laplace variable |
| $t$ | time |
| $u$ | velocity |
| $v_k$ | specific volume of phase-k |
| $v_{fg}$ | $v_g - v_f$ |
| $\langle x \rangle$ | flow quality |
| $z$ | axial position |

*Greek symbols*

| | |
|---|---|
| $\alpha$ | void fraction |
| $\Lambda_j$ | $1/2fL_j/D_H$ = friction loss number |
| $\xi$ | weighting factor |
| $\bar{\rho}$ | $\rho_g\alpha + \rho_f(1-\alpha)$ = mixture density |
| $\omega$ | angular frequency |

*Subscripts*

| | |
|---|---|
| f | liquid |
| g | vapor |
| i | inlet |
| e | exit |

*Other symbols*

| | |
|---|---|
| NVG | net vapor generation (point) |
| 1Φ | single-phase |
| 2Φ | two-phase |

(i.e., the Zuber number, $N_{Zu}$) and the subcooling number, $N_{sub}$, as the two most important non-dimensional parameters that define the stability of a boiling system. Saha [4] obtained comprehensive experimental results for a boiling channel. He determined the linear stability map for density-wave oscillations. Achard et al. [5] used a distributed parameter homogeneous equilibrium model (HEM) to perform linear stability and dynamic bifurcation analysis. Both supercritical and subcritical Hopf bifurcations were found. Lahey and Podowski [6] presented a general analysis of the dynamics of two-phase flows and illustrated this approach using a boiling channel. Clausse and Lahey [7] presented a nodal, lumped parameter, HEM model with moving nodal boundaries, based on a Galerkin nodal approximation of the conservation equations for a boiling channel, and, by numerically integrating the nodal equations, they found limit cycles and chaotic oscillations. Takenaka [8] and Chang et al. [9] expanded this model to simulate the dynamics of a BWR and an SBWR.

In this paper, a moving boundary nodal model is derived for the stability and bifurcation analysis of a heated boiling channel. The model is based on the technique used in fixed nodal models [10], where the nodal integral of the variables in the conservation equations are approximated by a weighted average of those variables evaluated at the boundaries of the nodes. In this model, the integration domain is time dependent, and this characteristic means that a much smaller number of nodes is needed.

## 2. Derivation of the model

In deriving the system of ordinary differential equations that represents the nodal model of a parallel boiling channel, the following assumptions have been made:

- The system pressure is constant
- The flow is one-dimensional
- The power is uniform in the axial direction
- Both phases are incompressible
- Subcooled boiling is included via Levy's profile-fit model [11]
- The phasic slip between the phases is quantified by drift-flux parameters, $C_0$ and $V_{gj}$ [12]
- Viscous dissipation, kinetic energy, potential energy and flow work are neglected in the energy equation
- The liquid inlet temperature is constant
- The heated channel is vertical

The non-dimensional, 1-D continuity, momentum and energy conservation equations for the two-phase mixture are [11]:

### 2.1. Mass

$$\frac{\partial u^+}{\partial z^+} = 0, \quad \text{single-phase region}$$

$$\frac{\partial \langle \bar{\rho} \rangle^+}{\partial t^+} + \frac{\partial G^+}{\partial z^+} = 0, \quad \text{two-phase region} \tag{1a}$$

*2.2. Momentum*

$$-\frac{\partial p^+}{\partial z^+} - \frac{1}{Fr}\langle\bar{\rho}\rangle^+ - \Lambda\frac{G^{+^2}}{\langle\bar{\rho}\rangle^+} - \sum_{\mathrm{i}} K_{\mathrm{i}}\delta(z^+ - z_{\mathrm{i}}^+)\frac{G_{\mathrm{i}}^{+^2}}{\langle\bar{\rho}\rangle_{\mathrm{i}}^+} - \frac{\partial}{\partial z^+}\left[\frac{(1-\langle\bar{\rho}\rangle^+)}{\left(\langle\bar{\rho}\rangle^+ - \frac{v_{\mathrm{f}}}{v_{\mathrm{g}}}\right)} \times \frac{\frac{v_{\mathrm{f}}}{v_{\mathrm{g}}}}{\langle\bar{\rho}\rangle^+}(V_{\mathrm{gj}}^{\prime +})^2\right] = \frac{\partial G^+}{\partial t^+} + \frac{\partial}{\partial z^+}\left(\frac{G^{+^2}}{\langle\bar{\rho}\rangle^+}\right) \tag{1b}$$

*2.3. Energy*

$$\frac{\partial h^+}{\partial t^+} + u_{\mathrm{i}}^+\frac{\partial h^+}{\partial z^+} = Q'''^+\frac{N_{\mathrm{sub}}}{N_{\mathrm{Zu}}}, \quad \text{single-phase region} \tag{1c}$$

The non-dimensional state equation is given by [11]:

$$\langle\bar{\rho}\rangle = 1, \quad \text{single-phase region}$$

$$\langle\bar{\rho}\rangle = 1 - \frac{\langle x\rangle}{C_0\left(1 + \langle x\rangle\frac{v_{\mathrm{g}}}{v_{\mathrm{fg}}}\right) + \frac{V_{\mathrm{gj}}^+}{G^+}\frac{v_{\mathrm{f}}}{v_{\mathrm{g}}}}, \tag{2a}$$

two-phase region

where,

$$\langle x\rangle = h^+\frac{v_{\mathrm{f}}}{v_{\mathrm{fg}}}N_{\mathrm{Zu}}, \quad \text{thermodynamic equilibrium}$$

$$\langle x\rangle = N_{\mathrm{Zu}}\left\{h^+\frac{v_{\mathrm{f}}}{v_{\mathrm{fg}}} - (h^+ - h_{\mathrm{NVG}}^+)\frac{v_{\mathrm{f}}}{v_{\mathrm{fg}}}\exp\left[\frac{h^+ - h_{\mathrm{NVG}}^+}{h_{\mathrm{NVG}}^+}\right]\right\}, \tag{2b}$$

subcooled boiling

Levy's profile-fit model has been used to model subcooled boiling [11].

The Zuber number (also called the phase-change number), $N_{\mathrm{Zu}}$, is defined as:

$$N_{\mathrm{Zu}} = \frac{q''_0 P_{\mathrm{H}} L_{\mathrm{H}}}{A_{x-\mathrm{s}}\rho_{\mathrm{f}} u_{\mathrm{i}0}}\frac{v_{\mathrm{fg}}}{h_{\mathrm{fg}} v_{\mathrm{f}}} \tag{3}$$

## 3. Nodalization criterion

Two regions can be identified in the boiling channel: a single-phase region, which extends from the inlet of the channel to the net vapor generation (NVG) point, and a two-phase region, which extends from the NVG point to the exit of the channel. The boiling channel is divided in $N = N_{1\Phi} + N_{2\Phi}$ nodes with moving boundaries, where $N_{1\Phi}$ and $N_{2\Phi}$ are the number of nodes in the single-phase and two-phase region, respectively. The boundaries of the nodes follow the axial position inside the channel where the enthalpy of the fluid (or of the liquid/vapor mixture in the two-phase region) reaches a determined value. In the single-phase region, the upper boundary of the $n$th node follows the position inside the channel where the non-dimensional enthalpy of the liquid is equal to $h_n^+ = h_{\mathrm{i}}^+ + n\Delta h_{1\Phi}^+$, where $h_{\mathrm{i}}^+$ is the non-dimensional inlet enthalpy and $\Delta h_{1\Phi}^+ = (h_{\mathrm{NVG}}^+ - h_{\mathrm{i}}^+)/N_{1\Phi}$ is the change in enthalpy in each single-phase ($1\Phi$) node.

Analogously, the upper boundary of the $n$th node in the two-phase region is defined as the axial position inside the channel where the enthalpy of the two-phase ($2\Phi$) mixture reaches the values $h_n^+ = h_{\mathrm{NVG}}^+ + (n - N_{1\Phi})\Delta h_{2\Phi}^+$, where, $\Delta h_{2\Phi}^+ = (h_{\mathrm{e}}^+ - h_{\mathrm{NVG}}^+)/N_{2\Phi}$, is the change in enthalpy in each two-phase node.

The derivation of the ODEs that describe the dynamics of the boiling channel is based on the integration of the spatial variable in the conservation equations. The conservation equation is integrated inside each node, $[L_{n-1}^+, L_n^+]$. The general form of this integral can be written as:

$$\int_{L_{n-1}^+}^{L_n^+}\frac{\partial y}{\partial t^+}\,\mathrm{d}z^+ + \int_{L_{n-1}^+}^{L_n^+}\frac{\partial F(z^+)}{\partial z^+}\,\mathrm{d}z^+ = \int_{L_{n-1}^+}^{L_n^+} H(z^+)\,\mathrm{d}z^+ \tag{4}$$

The integration domain is time dependent: thus Leibnitz's rule has to be applied to integrate the first term:

$$\frac{\mathrm{d}}{\mathrm{d}t^+}\int_{L_{n-1}^+}^{L_n^+} y\,\mathrm{d}z^+ - \frac{\mathrm{d}L_n^+}{\mathrm{d}t}y(L_n^+) + \frac{\mathrm{d}L_{n-1}^+}{\mathrm{d}t^+}y(L_{n-1}^+) + F(L_n^+) - F(L_{n-1}^+) = \int_{L_{n-1}^+}^{L_n^+} H(z^+)\,\mathrm{d}z^+ \tag{5}$$

The integrals can be written in terms of the nodal average of the integrand as:

$$\int_{L_{n-1}^+}^{L_n^+} f\,\mathrm{d}z^+ = (L_n^+ - L_{n-1}^+)\overline{f_n} \tag{6a}$$

where,

$$\overline{f_n} \equiv \xi_k f(L_n^+) + (1-\xi_k) f(L_{n-1}^+) \quad k = 1\Phi, 2\Phi \tag{6b}$$

The resulting ODE for the $n$th node is then:

$$\begin{aligned}&(L_n^+ - L_{n-1}^+)\xi_k \frac{\mathrm{d}y_n}{\mathrm{d}t^+} + (L_n^+ - L_{n-1}^+)(1-\xi_k)\frac{\mathrm{d}y_{n-1}}{\mathrm{d}t^+}\\ &\quad + (\overline{y_n} - y_n)\frac{\mathrm{d}L_n^+}{\mathrm{d}t^+} - (\overline{y_n} - y_{n-1})\frac{\mathrm{d}L_{n-1}^+}{\mathrm{d}t^+}\\ &\quad = (L_n^+ - L_{n-1}^+)\overline{H_n} + F(L_{n-1}^+) - F(L_n^+)\end{aligned} \tag{7}$$

## 4. Single-phase region

The single-phase region extends from the inlet of the channel to the net vapor generation point $L_{\mathrm{NVG}}$. The net vapor generation point (NVG) is defined as the axial position where subcooled boiling begins. The criterion used in the determination of the onset of subcooled boiling is given by the non-dimensional form of the modified Saha correlation [4] given by Clausse and Lahey [13]:

$$-h_{\mathrm{NVG}}^+ = \frac{q''^+ \Gamma Pe_0}{455}, \quad Pe < Pe_c$$

$$-h_{\mathrm{NVG}}^+ = \min\begin{cases}-h_{\mathrm{i}}^+\\ 154\dfrac{q''^+\Gamma}{u_{\mathrm{i}}^+}(1-\eta)\end{cases}, \quad Pe > Pe_c \tag{8a}$$

where,

$$Pe_c = 70{,}000(1 + 99St_{\mathrm{i}})^{-1} \tag{8b}$$

$$Pe_0 = \frac{\rho_{\mathrm{f}} u_{\mathrm{i}0} c_{\mathrm{pf}} D_{\mathrm{H}}}{k_{\mathrm{f}}} \tag{8c}$$

$$\Gamma = \frac{A_{\mathrm{x-s}}}{P_{\mathrm{H}} L_{\mathrm{H}}} \tag{8d}$$

$$\eta = \frac{99}{99 + \dfrac{1}{St_{\mathrm{i}}}} \tag{8e}$$

$$St_{\mathrm{i}} = -\frac{q''^+\Gamma}{u_{\mathrm{i}}^+ h_{\mathrm{i}}^+} \tag{8f}$$

To obtain the set of ODEs that describe the dynamics of the single-phase region, the procedure described in the preceding section has been applied to the single-phase energy conservation equation. The following ODE is obtained for the $n$th node:

$$\begin{aligned}&(1-\xi_{1\Phi})\frac{\mathrm{d}L_{n-1}^+}{\mathrm{d}t^+} + \xi_{1\Phi}\frac{\mathrm{d}L_n^+}{\mathrm{d}t^+} = u_{\mathrm{i}}^+ - \left(\bar{Q}_n^+ \frac{N_{\mathrm{sub}}}{N_{\mathrm{Zu}}\Delta h_{1\Phi}^+}\right.\\ &\quad \left. + \frac{n}{N_{1\Phi}}\frac{\mathrm{d}h_{\mathrm{NVG}}^+}{\mathrm{d}t^+}\right)(L_n^+ - L_{n-1}^+)\end{aligned} \tag{9}$$

In this set of equations, time is the remaining independent variable and the dependent variables are the inlet velocity and the position (i.e., the NVG point) of the boundaries of the nodes, $L_n^+$. The $N_{1\Phi}$th equation is the ODE for the boiling boundary, where $N_{1\Phi}$ should always be chosen to be an even number. The enthalpy of the NVG point will depend on the value of the Peclet number.

## 5. Two-phase region

The two-phase region extends from the NVG point and the exit of the channel. To obtain a set of ODEs for the dynamics of the two-phase region, the procedure for the integration of the conservation equations has been applied to the continuity equation of the two-phase mixture:

$$\begin{aligned}&(L_n^+ - L_{n-1}^+)\frac{\mathrm{d}}{\mathrm{d}t^+}\overline{\langle\bar{\rho}\rangle_n^+} + \left(\overline{\langle\bar{\rho}\rangle_n^+} - \langle\bar{\rho}\rangle_n^+\right)\frac{\mathrm{d}L_n^+}{\mathrm{d}t^+}\\ &\quad - \left(\overline{\langle\bar{\rho}\rangle_n^+} - \langle\bar{\rho}\rangle_{n-1}^+\right)\frac{\mathrm{d}L_{n-1}^+}{\mathrm{d}t^+}\\ &\quad = G_n^+ - G_{n-1}^+\end{aligned} \tag{10}$$

The average of the density can be approximated as in Eq. (5b):

$$\frac{\mathrm{d}\overline{\langle\bar{\rho}\rangle_n^+}}{\mathrm{d}t^+} = (1-\xi_{1\Phi})\frac{\mathrm{d}\langle\bar{\rho}\rangle_{n-1}^+}{\mathrm{d}t^+} + \xi_{2\Phi}\frac{\mathrm{d}\langle\bar{\rho}\rangle_n^+}{\mathrm{d}t^+} \tag{11}$$

and the time derivative of the density is given by:

$$\frac{\mathrm{d}\langle\bar{\rho}\rangle_n^+}{\mathrm{d}t^+} = \frac{C_0(1-\langle\bar{\rho}\rangle_n^+)}{C_0\left[h_n^+ + h_{\mathrm{NVG}}^+\ \exp\left(\dfrac{h_n^+}{h_{\mathrm{NVG}}^+}-1\right)+1\right]+\dfrac{V_{\mathrm{gj}}^+}{G_n^+}}$$

$$\times\left\{\left[1-\ \exp\left(\frac{h_n^+}{h_{\mathrm{NVG}}^+}-1\right)\right]\frac{\mathrm{d}h_n^+}{\mathrm{d}t^+}\right.$$

$$-\ \exp\left(\frac{h_n^+}{h_{\mathrm{NVG}}^+}-1\right)\left(1-\frac{h_n^+}{h_{\mathrm{NVG}}^+}\right)\frac{\mathrm{d}h_{\mathrm{NVG}}^+}{\mathrm{d}t^+}$$

$$-\frac{C_0\dfrac{V_{\mathrm{gj}}^+}{G_n^{+^2}}(1-\langle\bar{\rho}\rangle_n^+)}{C_0(1-\langle\bar{\rho}\rangle_n^+)-1}\rho_{h_n}^+\frac{\mathrm{d}u_{\mathrm{i}}^+}{\mathrm{d}t^+}$$

$$+\frac{C_0\dfrac{V_{\mathrm{gj}}^+}{G_n^{+^2}}(1-\langle\bar{\rho}\rangle_n^+)}{C_0(1-\langle\bar{\rho}\rangle_n^+)}\rho_{h_n}^+$$

$$\left.\hat{N}_{\mathrm{pch}}\sum_{j=k}^{n}\bar{Q}_j^+\left(\frac{\mathrm{d}L_j^+}{\mathrm{d}t^+}-\frac{\mathrm{d}L_{j-1}^+}{\mathrm{d}t^+}\right)\right\} \tag{12}$$

where the $k$th node is the first node in the two-phase region where the enthalpy becomes positive. Substituting into Eq. (10) yields a nodal ODE for the dynamics of the two-phase region:

$$a(n,N_{1\Phi}+N_{2\Phi})\frac{\mathrm{d}h_{\mathrm{e}}^+}{\mathrm{d}t^+}+a(n,N_{1\Phi}+N_{2\Phi}+1)\frac{\mathrm{d}u_{\mathrm{i}}^+}{\mathrm{d}t^+}$$

$$+a(n,N_{1\Phi})\frac{\mathrm{d}L_{\mathrm{NVG}}^+}{\mathrm{d}t^+}+a(n,n)\frac{\mathrm{d}L_n^+}{\mathrm{d}t^+}$$

$$=G_{n-1}^+-G_n^+ \tag{13}$$

where the coefficients of the time derivatives are:

$$a(n,N_{1\Phi}+N_{2\Phi})$$

$$=(L_n^+-L_{n-1}^+)\left\{\frac{\dfrac{n-N_{1\Phi}}{N_{2\Phi}}(1-\xi_{2\Phi})}{\left[C_0(1+h_n^+)+\dfrac{V_{\mathrm{gj}}^+}{G_n^+}\right]^2}\right.$$

$$\times\left[-C_0-\frac{V_{\mathrm{gj}}^+}{G_n^+}\left(1-\frac{n-N_{1\Phi}}{N_{2\Phi}}\rho_{h_n}^+\right)\right]$$

$$\times\frac{\dfrac{n-1-N_{1\Phi}}{N_{2\Phi}}\xi_{2\Phi}}{\left[C_0(1+h_n^+)+\dfrac{V_{\mathrm{gj}}^+}{G_{n-1}^+}\right]^2}$$

$$\left.\times\left[-C_0-\frac{V_{\mathrm{gj}}^+}{G_{n-1}^+}\left(1-\frac{n-1-N_{1\Phi}}{N_{2\Phi}}\rho_{h_{n-1}}^+\right)\right]\right\} \tag{14a}$$

## 6. Closure

At this point, there is one more unknown than there are ODEs. Closure can be derived by integrating the momentum equation in space for the heated channel:

$$-\int_0^1\langle\bar{\rho}\rangle^+\,\mathrm{d}z^+-\int_0^1\Lambda\frac{G^{+^2}}{\langle\bar{\rho}\rangle^+}\,\mathrm{d}z^-$$

$$+\sum_{\mathrm{i}}\int_0^1K_{\mathrm{i}}\delta(z^+-z_{\mathrm{i}}^+)\frac{G^{+^2}}{\langle\bar{\rho}\rangle^+}\,\mathrm{d}z^+$$

$$-\int_0^1\frac{\partial}{\partial z^+}\left(\frac{1-\langle\bar{\rho}\rangle^+}{\langle\bar{\rho}\rangle^+-\dfrac{v_{\mathrm{f}}}{v_{\mathrm{g}}}}\frac{v_{\mathrm{f}}}{v_{\mathrm{g}}}\frac{1}{\langle\bar{\rho}\rangle^+}(V_{\mathrm{gj}}^{+'})^2\right)\mathrm{d}z^+ \tag{15}$$

$$=\int_0^1\frac{\partial G^+}{\partial t^+}\,\mathrm{d}z^++\int_0^1\frac{\partial}{\partial z^+}\left(\frac{G^{+^2}}{\langle\bar{\rho}\rangle^+}\right)\mathrm{d}z^+$$

which yields the following equation:

$$Eu-\Delta p_{\mathrm{g}}^+-\Delta p_{\mathrm{lf}}^+-\Delta p_{\mathrm{f}}^+-\Delta p_{\mathrm{df}}^+-\Delta p_{\mathrm{a}}^+$$

$$=\sum_{n=1}^{N_{1\Phi}}a(n,N+1)\frac{\mathrm{d}L_n^+}{\mathrm{d}t^+}$$

$$+\sum_{n=N_{1\Phi}+1}^{N-1}a(n,N+1)\frac{\mathrm{d}L_n^+}{\mathrm{d}t^+}+a(N,N+1)\frac{\mathrm{d}h_{\mathrm{e}}^+}{\mathrm{d}t^+} \tag{16}$$

$$+a(N+1,N+1)\frac{\mathrm{d}u_{\mathrm{i}}^+}{\mathrm{d}t^+}$$

where, $N = N_{1\Phi} + N_{2\Phi}$; $\Delta p_{\rm g}^{+}$ is the gravity pressure loss; $\Delta p_{\rm lf}^{+}$ is the localized hydraulic loss; $\Delta p_{\rm f}^{+}$ is the distributed friction loss; $\Delta p_{\rm df}^{+}$ is the drift-flux term; $\Delta p_{\rm a}^{+}$ is the spatial acceleration term. These terms are given by:

$$\Delta p_{\rm g}^{+} = L_{\rm NVG}^{+} - b\hat{N}_{\rm pch} \sum_{n=1}^{N_{1\Phi}} \overline{h_n^{+}}(L_n^{+} - L_{n-1}^{+}) + \sum_{n=N_{1\Phi}}^{N} \overline{\langle\bar{\rho}\rangle_n^{+}}(L_n^{+} - L_{n-1}^{+}) \tag{17a}$$

where the Boussinesq approximation [8] has been used to approximate the dependence of the liquid density on enthalpy in the single-phase region.

$$\Delta p_{\rm lf}^{+} = \sum_{\rm i} K_{\rm i} \frac{G_{\rm i}^{+^2}}{\langle\bar{\rho}\rangle_{\rm i}^{+}} = K_{\rm i} u_{\rm i}^{+^2} + K_{\rm e} \frac{G_{\rm e}^{+^2}}{\langle\bar{\rho}\rangle_{\rm e}^{+}} \tag{17b}$$

where the localized hydraulic losses are assumed to be concentrated at the inlet and exit of the heated channel.

$$\Delta p_{\rm f}^{+} = \Lambda_{1\Phi} u_{\rm i}^{+^2} L_{\rm NVG}^{+} + \sum_{n=N_{1\Phi}+1}^{N_{1\Phi}+N_{2\Phi}} \Lambda_{2\Phi} \left[ (1-\xi_{2\Phi}) \frac{G_{n-1}^{+^2}}{\langle\bar{\rho}\rangle_{n-1}^{+}} + \xi_{2\Phi} \frac{G_n^{+^2}}{\langle\bar{\rho}\rangle_n^{+}} \right] \times (L_n^{+} - L_{n-1}^{+}) \tag{17c}$$

$$\Delta p_{\rm df}^{+} = \frac{1 - \langle\bar{\rho}\rangle_{\rm e}^{+}}{\langle\bar{\rho}\rangle_{\rm e}^{+} - \frac{v_{\rm f}}{v_{\rm g}}} \frac{v_{\rm f}}{v_{\rm g}} \frac{1}{\langle\bar{\rho}\rangle_{\rm e}^{+}} (V_{\rm gj}^{\prime +})^2 \tag{17d}$$

$$\Delta p_{\rm a}^{+} = u_{\rm i}^{+^2} - \frac{G_{\rm e}^{+^2}}{\langle\bar{\rho}\rangle_{\rm e}^{+}} \tag{17e}$$

The integration of the inertia term yields the following coefficients for the time derivatives:

$$a(N+1, n) = \hat{N}_{\rm pch} \bar{Q}_n^{+} \rho_{h_n}^{+} + G_n^{+} \tag{18a}$$

$$a(N+1, N_{1\Phi}) = (1-\xi_{2\Phi}) \hat{N}_{\rm pch} \bar{Q}_{N_{1\Phi}}^{+} + u_{\rm i}^{+} \tag{18b}$$

$$a(N+1, N) = \sum_{n=N_{1\Phi}+1}^{N} \left[ (1-\xi_{2\Phi}) \rho_{h_{n-1}}^{+} \frac{n-1}{N_{2\Phi}} G_{n-1}^{+} + \xi_{2\Phi} \rho_{h_n}^{+} \frac{n}{N_{2\Phi}} G_n^{+} \right] \tag{18c}$$

## 7. Linear stability analysis

The system of nonlinear ODEs obtained with the generalized nodal model can be written in matrix form as,

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \tag{19a}$$

where,

$$\underline{x} \equiv (L_1^{+}, \ldots, L_{N_{1\Phi-1}}^{+}, L_{\rm NVG}^{+}, \ldots, L_{N_{1\Phi}+N_{2\Phi}}^{+}, h_{\rm e}^{+}, \ldots, h_{N_{\rm R}}^{+}, \ldots, u_{\rm i}^{+})^{\rm T} \tag{19b}$$

is the matrix vector of dependent variables.

The system of nonlinear ODEs was linearized around its fixed points and the eigenvalues of its Jacobian matrix were calculated. The linear stability of the system can be determined from the eigenvalues, $\lambda_{\rm i}$, of the Jacobian matrix. If all the eigenvalues, $\lambda_{\rm i}$, have negative real parts, the system is linearly stable. This means, that if a small perturbation is applied to the system, it will go back to the stable fixed point, or steady state. If an eigenvalue is real and positive, the system has an excursive instability, (i.e., a small perturbation applied to the system in the steady state will cause a divergence away from this unstable fixed point). If any pair of complex conjugate eigenvalues have positive real parts, the system is linearly unstable and oscillatory (i.e., an unstable phase-plane spiral). This means that a small perturbation applied to the system in the unstable steady state will cause a divergence away from the steady state in an oscillatory fashion.

The eigenvalues of the system were calculated numerically for different numbers of axial nodes and for different values of $\xi_{1\Phi}$ and $\xi_{2\Phi}$, and the linear stability boundary plotted in the $N_{\rm zu}$–$N_{\rm sub}$ phase plane. Fig. 1 shows the linear stability boundary between a $D_0$ region, where the system is linearly stable, and the $D_2$ region, where the system has a periodic oscillatory response, for $\xi_{1\Phi} = \xi_{2\Phi} = 0.5$, and for different number of nodes in the single-phase region of a boiling channel. It is apparent from the figure that increasing the number of nodes leads to a better agreement with the experimental results, and nodal convergence was essentially achieved for $N_{1\Phi} = 6$ nodes.

## 8. Hopf bifurcation analysis

Nonlinear oscillatory responses have been observed in nuclear reactors. Since oscillations can lead to operational situations where the safety of the reactor may be compromised, the ability to predict them is necessary. A system of ODEs is said to go through a Hopf

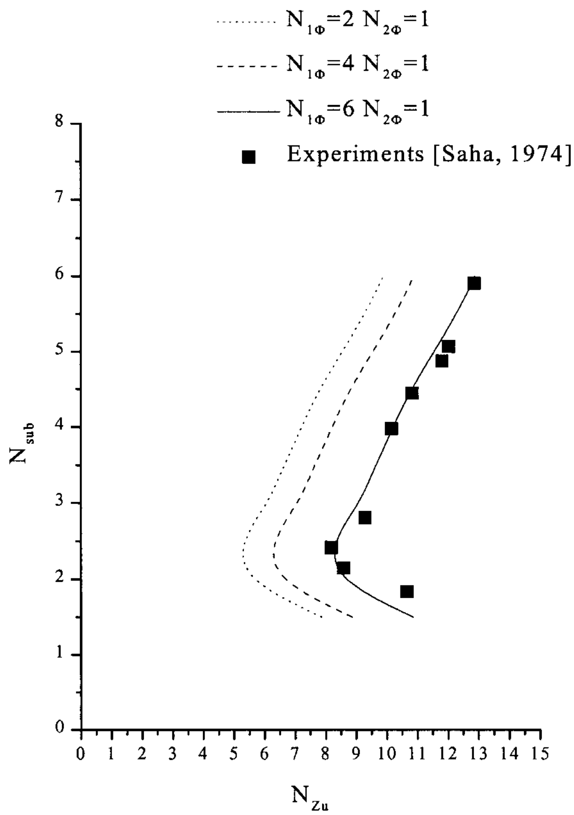


Fig. 1. Linear stability map. Effect of the number of nodes.

bifurcation at a fixed point ($\underline{x}_0$, $\underline{\mu}_0$) if [14]:

- The Jacobian matrix of the system at the fixed point, $\underline{\underline{J}}(\underline{x}_0, \underline{\mu}_0)$, has a pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.
- $d/d\mu_k \; Re\lambda(\mu_k)) \mid_{\underline{\mu}=\underline{\mu}_0} \neq 0$, where $\underline{\mu}_0$ is a vector of parameters, and $\mu_k$ is an element of that vector, selected as the bifurcation parameter.

The bifurcation analysis was performed by adding a subroutine with the ODEs derived herein, using the nodal method, to the computer program AUTO [15].

Fig. 2 shows a supercritical Hopf bifurcation. In this calculation, the subcooling number, $N_{sub}$, the Euler number, $Eu$, the distribution friction number, $\Lambda$, and the inlet and exit flow restriction coefficients were maintained constant and equal to their value at the linear stability boundary. The bifurcation parameter, $\mu$, was defined as:

$$\mu = N_{Zu} - N_{Zu_0} \tag{20}$$

where the subscript 0 denotes the value of the parameter at the linear stability boundary.

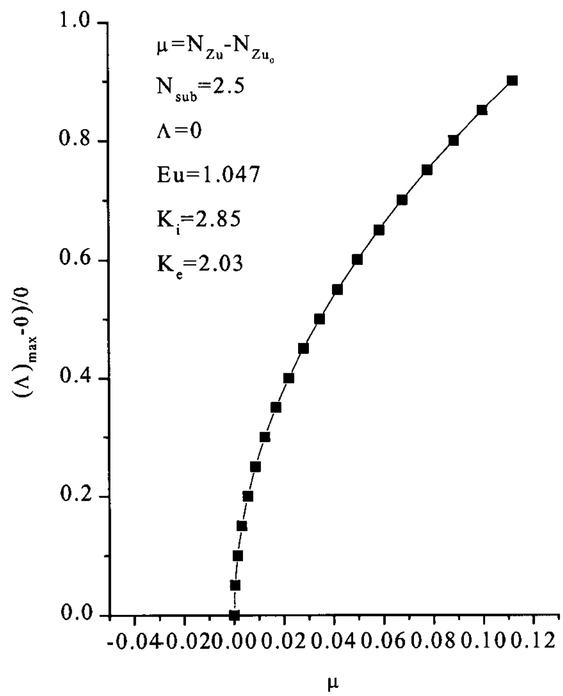


Fig. 2. A typical supercritical Hopf bifurcation for a boiling channel.

Unstable limit cycles were found in the regions of the parameter space where previous works found the same response [5]. Fig. 3 presents a typical subcritical Hopf bifurcation for $K_i = K_e = 0.0$. In this situation, if the system is perturbed with an amplitude larger than the amplitude of the unstable limit cycle (dashed line), the solution diverges. In contrast, if the amplitude of the perturbation is smaller than the maximum of the unstable limit cycle, the system evolves to the stable fixed point. This implies a region in parameter space where a perturbation with a large enough amplitude may cause an excursive instability in a linearly stable system.

## 9. Frequency domain analysis

Since the set of ODEs for the forced single-phase region are linear, classical control theory can be applied to analyze the frequency response of the system as a function of the number of nodes and compare it with the frequency response of the system defined by the transfer function of the inlet velocity as input and the time evolution of the boiling boundary as output [16]. Assuming saturated boiling, the equation for the $n$-th node is given by:

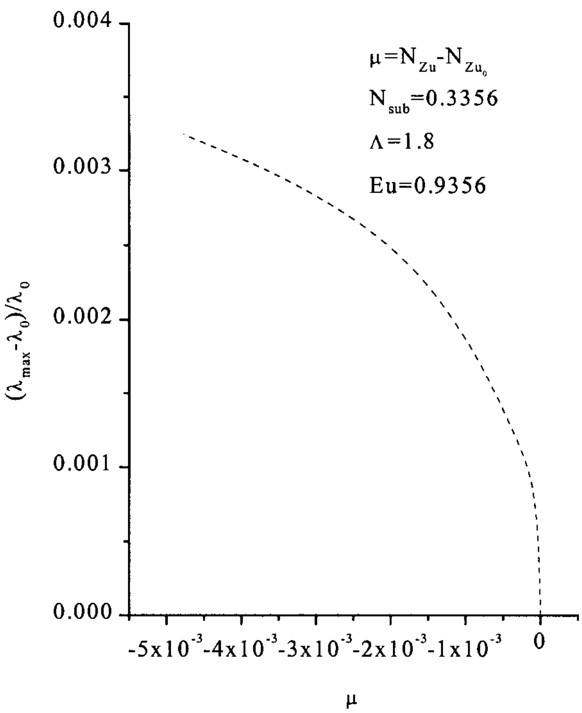


Fig. 3. A typical subcritical Hopf bifurcation for a boiling channel.

$$\xi_{1\Phi}\frac{dL_{n-1}^{+}}{dt^{+}}+(1-\xi_{1\Phi})\frac{dL_{n}^{+}}{dt^{+}}=u_{i}^{+}-a(L_{n}^{+}-L_{n-1}^{+}) \tag{21}$$

where, $a = Q^{+}N_{Zu}/N_{sub}$.

Laplace transforming both sides of the equation yields:

$$s\xi_{1\Phi}L_n(s)+s(1-\xi_{1\Phi})L_{n-1}(s)-U(s)-aL_n(s)+aL_{n-1}(s) \tag{22}$$

where the indication of non-dimensional quantity $(+)$ has been dropped for simplicity, and it has been assumed that the arbitrary initial condition for the boundaries of all the nodes is equal to zero. Solving for $L_n(s)$ and back-substituting $L_{n-1}$ results in the following expression:

$$L_n(s)=\left\{\frac{U(s)}{(-a+\xi_{1\Phi}s)}-\frac{U(s)(a+(1-\xi_{1\Phi})s)}{(-a+\xi_{1\Phi}s)^2}+\frac{U(s)(a+(1-\xi_{1\Phi})s)^{n-1}(-1)^{n-1}}{(-a+\xi_{1\Phi}s)^n}+\cdots\right\} \tag{23}$$

The transfer function, $G$, of a linear system is defined as the ratio between the output and the input (in this case, the boiling boundary and the inlet velocity, respectively), therefore:

$$G(s)=\frac{L_{N_{1\Phi}}(s)}{U(s)}=\sum_{k=1}^{N_{1\Phi}}\frac{(a+(1-\xi_{1\Phi})s)(-1)^{k-1}}{(-a+\xi_{1\Phi}s)^k}=\frac{\left(\frac{s(1-\xi_{1\Phi})+a)}{-\xi_{1\Phi}s+a}\right)^{N_{1\Phi}+1}(-\xi_{1\Phi}s+a)}{s[(1-\xi_{1\Phi})s+a]}-\frac{1}{s} \tag{24}$$

The modulus of this transfer function gives the gain of the system.

The exact solution for the boiling boundary's dynamics is given in Eq. (25).

$$N(t)\equiv L_{N_{1\Phi}^{(t)}}=\int_{t-v}^{t}u_i(t')\,dt' \tag{25}$$

where, $v = N_{sub}/N_{Zu}Q^{+}$ is the non-dimensional time required for a particular control volume to loose its subcooling [11].

This equation can be Laplace-transformed to obtain the exact transfer function for the system with a periodic inlet velocity as forcing function, $u_i = u_{i_0} + b\ \sin(\omega t)$:

$$G_{exact}(s)=\frac{L_{N_{1\Phi}}(s)}{U(s)}=\frac{u_{i_0}\omega v(1+\omega^2)-b+bs[s\ \cos(\omega v)+w\ \sin(\omega v)]}{\omega s(\omega^2+s^2)} \tag{26}$$

The modulus of this function gives the gain of the exact solution as:

$$|\ G_{exact}(s=j\omega)\ |=\frac{\sqrt{1(1-\ \cos(\omega v))}}{\omega} \tag{27}$$

Fig. 4(a, b) shows the comparison between the gain of the nodal model and the exact solution, for different numbers of nodes and $\xi_{1\Phi}=0.5$ and $\xi_{1\Phi}=1.0$, respectively. Fig. 4(a) shows a better agreement with the frequency response of the exact solution. It should be noted that the equations obtained for $\xi_{1\Phi}=0.5$ coincide with the equations obtained by integrating the single-phase energy equations assuming a linear enthalpy profile [7], and the set of equations obtained with $\xi_{1\Phi}=1.0$ is the same as the equations derived with a backward-difference finite difference approximation. It can also be observed that increasing the number of nodes leads to a larger number of minima for the nodal model and hence a better agreement with the exact solution. These results also show that the number of nodes must increase with frequency. The main drawback is that for chaotic predictions, virtually

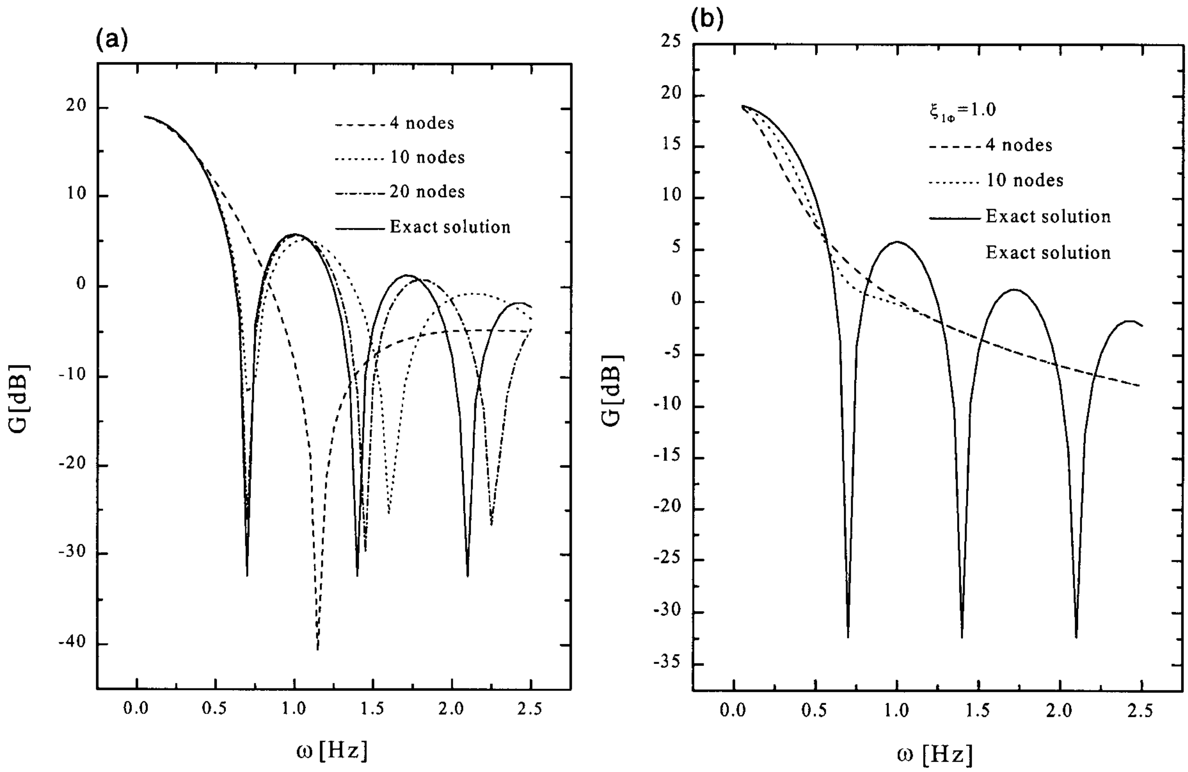


Fig. 4. Gain vs frequency for the system of ODEs and the exact solution. Effect of the number of nodes $= \xi_{1\Phi} = 0.5$ (a); $\xi_{1\Phi} = 1.0$ (b).

all frequencies are present and therefore an infinite number of nodes would be needed. Unfortunately, this is a fundamental problem when any finite-difference/element scheme is used to analyze chaotic phenomena.

## 10. Summary and conclusions

A generalized moving boundary nodal model has been derived for the stability analysis of boiling systems. This model agrees well with experiments and with results obtained with previous fixed-node and distributed-parameter models. The model predicts the linear stability of a boiling system with a fairly small number of nodes, and weighting factors of $\xi_{1\Phi} = \xi_{2\Phi} = 0.5$ appear to give the best results, provided, of course, that an even number of axial nodes ($N_{1\Phi}$) are used in the single-phase part of the heated channel.

The comparison between the gain of the system of ODEs and the gain of the exact solution provides a convenient tool to estimate the number of single-phase nodes necessary to capture the behavior of the boiling boundary for a given cutoff frequency. This analysis shows, in particular, that care should be exercised interpreting the results when a large bandwidth (i.e., chaotic) response is obtained.

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